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# Commuting quantum traces: the case of reflection algebras

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## Abstract

We formulate a systematic construction of commuting quantum traces for reflection algebras. This is achieved by introducing two dual sets of generalized reflection equations with associated consistent fusion procedures. Products of their respective solutions yield commuting quantum traces.

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## 1. Introduction

The concept of quantum traces which will be discussed here goes back to the work of Maillet [1] where commuting quantum analogues of the classical Poisson-commuting traces of powers of Lax matrices  $\text{Tr} L^n$ ,  $\{n \in \mathbf{N}\}$  were explicitly constructed in the context of quantum group structures. Starting from the well-known fundamental quantum group relation

$$R_{12}(\lambda_1 - \lambda_2)L_{1q}(\lambda_1)L_{2q}(\lambda_2) = L_{2q}(\lambda_2)L_{1q}(\lambda_1)R_{12}(\lambda_1 - \lambda_2) \quad (1.1)$$

where the quantum  $R$  matrix obeys the Yang–Baxter equation [2, 3],

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1 - \lambda_2) \quad (1.2)$$

(and  $q$  denotes the quantum space<sup>3</sup> on which the generators of the quantum group inside the Lax matrix act), quantum commuting objects were built with the generic form

$$H_N = \text{Tr}_{1,\dots,N}(\mathcal{R}_{1,\dots,N}L_{1q}L_{2q}, \dots, L_{Nq}). \quad (1.3)$$

$\mathcal{R}$  is any auxiliary-space operator which satisfies a commuting form of the Yang–Baxter equation: precisely it is required that  $\mathcal{R}_{12,\dots,N}$  commute with coproduct-like structures of the form  $R_{1a}R_{2a}, \dots, R_{Na}$ . In particular, the choice  $\mathcal{R}_{12,\dots,N} = P_{12}R_{12}P_{23}R_{23}, \dots$ ,

<sup>3</sup> The quantum space may have the structure of a tensor product of single quantum spaces.

$P_{N-1N}R_{N-1N}$ , with  $P_{ab}$  the permutation operator acting on the auxiliary spaces  $V_a \otimes V_b$ , leads to the exact quantum analogue of the classical traces  $\text{Tr}(L^N)$ . A similar possibility occurs here justifying the characterization of our operators as ‘quantum traces’. This concept is different from the one leading to monodromy matrices, whose classical limit instead yields  $(\text{Tr } L)^N$  and powers thereof. We shall give a brief account of the construction in [1] as a preliminary illustration of the mechanisms involved in our more complicated procedure.

We approach here the problem of formulating a quantum trace construction for the more general quadratic algebras, discussed e.g. in [4], of the form

$$A_{12}T_1B_{12}T_2 = T_1C_{12}T_2D_{12} \quad (1.4)$$

where  $T$  encapsulates generators of the algebra and  $A, B, C, D$  are  $c$ -number structure matrices.

We shall focus here on a specific set of quadratic algebras known as ‘reflection algebras’ [5] where the matrices  $A, B, C, D$  are related to a single  $R$  matrix, albeit *a priori* depending on one complex spectral parameter. This structure is defined as [5, 6]:

$$R_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)T_2(\lambda_2) = T_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)T_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2). \quad (1.5)$$

$T_i(\lambda_i)$  is understood as a matrix acting on a finite-dimensional auxiliary space  $V_i$  with matrix entries being operators, representing the quantum reflection algebra, acting on some Hilbert space of quantum states hereafter denoted by the single label  $q$ .

Treatment of the most general case will be left for a forthcoming investigation [7] but it must be emphasized here that the particular case of reflection algebras actually embraces the full algebraic richness of the general case. Explicit realizations of quantum  $T$  matrices may easily be built using the fundamental results in [6] from quantum Lax matrices  $L_q(\lambda)$  obeying (1.2) and quantum reflection matrices  $K(\lambda)$  obeying (1.5) such as were constructed in [8] (see also references therein) as

$$T_q(\lambda) = L_q(\lambda)K(\lambda)(L_q(\lambda))^{-1}. \quad (1.6)$$

Note that the  $K$  matrix may act on some extra boundary quantum space  $q_b$  along the lines described in [9, 10].

Furthermore it is known [6, 4] that

$$t_{qq'}(\lambda) = \text{Tr}_1(K_{1q'}^+(\lambda)T_{1q}(\lambda)) \quad (1.7)$$

realize a family of mutually commuting operators on the quantum Hilbert space  $q \otimes q'$ :

$$[t_{qq'}(\lambda), t_{qq'}(\mu)] = 0 \quad (1.8)$$

where the matrix  $K_{q'}^+$  is a solution of the so-called ‘dual’ reflection equation [4, 12]:

$$\begin{aligned} R_{12}(-\lambda_1 + \lambda_2)K_1^+(\lambda_1)M_1^{-1}R_{21}(-\lambda_1 - \lambda_2 - 2\rho)M_1K_2^+(\lambda_2) \\ = K_2^+(\lambda_2)M_1R_{12}(-\lambda_1 - \lambda_2 - 2\rho)M_1^{-1}K_1^+(\lambda_1)R_{21}(-\lambda_1 + \lambda_2). \end{aligned} \quad (1.9)$$

Again  $K_i^+$  is understood as a matrix acting on the same finite-dimensional auxiliary space  $V_i$  with matrix entries being generically operators, representing the quantum reflection algebra, acting on a different Hilbert space of quantum states denoted by the single label  $q'$ . Furthermore the  $K^+$  matrix may act on some boundary quantum space denoted by the index  $q'_b$ . This notion of duality will be one crucial ingredient of our construction.

The problem is to obtain extensions of (1.7) as traces involving products (as opposed to *tensor* products) of more than one  $T$  (monodromy) matrix. Results on quantum traces obtained for algebra (1.5) will soon be extended [7] to the generic quadratic Yang–Baxter algebras by a suitable generalization of the procedure described here.

Throughout this paper we shall impose several conditions on the  $R$  matrix. We assume that

1. It obeys the Yang–Baxter equation (1.2).
2. It obeys the following symmetry requirement:

$$R_{12}(\lambda) = R_{21}(\lambda)^{t_1 t_2}. \quad (1.10)$$

For instance Yangian matrices constructed in [8] obey this condition.

3. It obeys unitarity and crossing relations

$$R_{12}(\lambda)R_{21}(-\lambda) \propto 1 \quad R_{12}(\lambda) = V_1 R_{12}^{t_2}(-\lambda - \rho) V_1 = V_2^{t_2} R_{12}^{t_1}(-\lambda - \rho) V_2^{t_2} \quad (1.11)$$

using (1.10) to obtain the second crossing relation.

Furthermore the  $R$  matrix obeys the crossing-unitarity relation

$$R_{21}(\lambda)^{t_1} M_1^{-1} R_{12}(-\lambda - 2\rho)^{t_1} \quad M_1 \propto 1 \quad (1.12)$$

and the commutation relation

$$[R_{12}, M_1 M_2] = 0. \quad (1.13)$$

Here  $V$  is a  $c$ -number matrix such that  $V^2 = 1$  and  $V^t V = M$ . The crossing used in [11] is of such type, and the Yangian  $R$  matrices constructed in [8] obey these conditions.

These conditions are particular formulations, in the context of reflection algebras, of more generic requirements for the structural matrices of quadratic Yang–Baxter algebras (1.4). Let us immediately indicate that general consistency 3-space exchange conditions generalizing (1.2) have already been formulated, see e.g. equations (15) in [4].

Our results are formulated in two basic steps:

1. We introduce two sets of dual generalized reflection equations extending (1.5), (1.9), closely related to the fusion procedure described in e.g. [12]. We establish the existence and the form of the simplest solutions in two basic lemmas. More general solutions may then be constructed using a dressing procedure.
2. We then build quantum traces for reflection algebras by combining solutions of the two dual sets of reflection equations in a form similar to (1.7) with suitable non-trivial dressing operators.

We finally conclude with some comments on the perspectives opened by our results.

## 2. Commuting traces

### 2.1. Review on Yang–Baxter-type algebras

Before we describe the construction of commuting traces associated with reflection algebras, we first would like to present the construction of quantum traces associated with Yang–Baxter algebras, realized in [4], from a slightly different point of view. Namely, we shall introduce a set of generalized fundamental equations with the help of which we will be able to build the commuting traces, and which we will use later in reflection algebras.

Let us first fix some convenient notation based on the coproduct structure of equation (1.1) [13, 14]. We introduce the objects, hereafter denoted as ‘fused’  $R$  matrices

$$\begin{aligned} R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} - \lambda_{\mathcal{M}'}) &= R_{11'}(\lambda_1 - \lambda_{1'}) R_{21'}(\lambda_2 - \lambda_{1'}) \dots R_{n1'}(\lambda_n - \lambda_{1'}) \\ &\quad \times R_{12'}(\lambda_1 - \lambda_{2'}) R_{22'}(\lambda_2 - \lambda_{2'}) \dots R_{n2'}(\lambda_n - \lambda_{2'}) \dots \\ &\quad \times R_{1m'}(\lambda_1 - \lambda_{m'}) R_{2m'}(\lambda_2 - \lambda_{m'}) \dots R_{nm'}(\lambda_n - \lambda_{m'}) \end{aligned} \quad (2.1)$$

where one defines ordered sets  $\mathcal{N} \equiv \langle 12 \dots n \rangle$ ,  $\mathcal{M}' \equiv \langle 1'2' \dots m' \rangle$ . We should emphasize that the notation we use here is essentially inspired by that introduced in [12] describing the fusion procedure for open spin chains. The above object satisfies the following properties:

$$R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} - \lambda_{\mathcal{M}'})^{t_{\mathcal{M}'\mathcal{N}}} = R_{\bar{\mathcal{M}}'\bar{\mathcal{N}}}(\lambda_{\bar{\mathcal{N}}} - \lambda_{\bar{\mathcal{M}'}}) \quad (2.2)$$

and the generalization of crossing (1.11),

$$R_{\bar{\mathcal{N}}\mathcal{M}'}(\lambda) = V_{\mathcal{N}} R_{\mathcal{N}\mathcal{M}'}(-\lambda - \rho)^{t_{\mathcal{M}'}} V_{\mathcal{N}} \quad R_{\mathcal{N}\bar{\mathcal{M}'}}(\lambda) = V_{\mathcal{M}'}^{t_{\mathcal{M}'}} R_{\mathcal{N}\mathcal{M}'}(-\lambda - \rho)^{t_{\mathcal{N}}} V_{\mathcal{M}'}. \quad (2.3)$$

It also satisfies unitarity and crossing unitarity, i.e.

$$\begin{aligned} R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} - \lambda_{\mathcal{M}'}) R_{\bar{\mathcal{M}}'\bar{\mathcal{N}}}(\lambda_{\bar{\mathcal{M}'}} - \lambda_{\bar{\mathcal{N}}}) &\propto 1 \\ R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} + \lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} M_{\mathcal{M}'}^{-1} R_{\bar{\mathcal{M}}'\bar{\mathcal{N}}}(-\lambda_{\bar{\mathcal{M}'}} - \lambda_{\bar{\mathcal{N}}} - 2\rho)^{t_{\mathcal{M}'}} M_{\mathcal{M}'} &\propto 1 \end{aligned} \quad (2.4)$$

and generalized commutation relation

$$[R_{\mathcal{N}\mathcal{M}'}, M_{\mathcal{N}} M_{\mathcal{M}'}] = 0. \quad (2.5)$$

Here one defines anti-ordered sets  $\bar{\mathcal{M}}' \equiv \langle m'(m' - 1) \dots 1' \rangle$ ,  $\bar{\mathcal{N}} \equiv \langle n(n - 1) \dots 1 \rangle$ . Now we introduce the set of generalized fundamental equations:

$$R_{\mathcal{N}\bar{\mathcal{M}'}}(\lambda_{\mathcal{N}} - \lambda_{\bar{\mathcal{M}'}}) \mathcal{L}_{\mathcal{N}}(\lambda_{\mathcal{N}}) \mathcal{L}_{\mathcal{M}'}(\lambda_{\mathcal{M}'}) = \mathcal{L}_{\mathcal{M}'}(\lambda_{\mathcal{M}'}) \mathcal{L}_{\mathcal{N}}(\lambda_{\mathcal{N}}) R_{\mathcal{N}\bar{\mathcal{M}'}}(\lambda_{\mathcal{N}} - \lambda_{\bar{\mathcal{M}'}}). \quad (2.6)$$

All solutions  $\mathcal{L}_{\mathcal{N}}$  of the above equation are actually good candidates for the construction of quantum commuting traces. In particular, the objects

$$H_{\mathcal{N}} = \text{Tr}_{\mathcal{N}} \mathcal{L}_{\mathcal{N}}(\lambda_{\mathcal{N}}) \quad (2.7)$$

realize a family of commuting operators  $[H_{\mathcal{N}}, H_{\mathcal{M}'}] = 0$ . An obvious solution of equation (2.6) is

$$\mathcal{L}_{\mathcal{N}}(\lambda_{\mathcal{N}}) = L_1(\lambda_1) \dots L_n(\lambda_n) \quad (2.8)$$

where  $L_i$  are Lax matrices obeying the fundamental equation (1.1). Obviously this solution leads to trivially decoupled traces, therefore one defines ‘dressed’ solutions of the generalized fundamental equation (see [1, 4]), for instance

$$\mathcal{L}_{\mathcal{N}}(\lambda_{\mathcal{N}}) = \check{R}_{12}(\lambda_1 - \lambda_2) \check{R}_{23}(\lambda_2 - \lambda_3) \dots \check{R}_{n-1n}(\lambda_{n-1} - \lambda_n) L_1(\lambda_1) \dots L_n(\lambda_n) \quad (2.9)$$

where  $\check{R}_{12} \equiv P_{12} R_{12}$  and  $P_{12}$  are the operators exchanging auxiliary spaces 1 and 2 and spectral parameters  $\lambda_1$  and  $\lambda_2$  (this last property is not easy to actually realize and we shall comment later on the practical aspects of this realization). The objects  $\check{R}$  are characterized by their commutation relations with fused  $R$ -matrices:

$$\check{R}_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1 - \lambda_3) R_{23}(\lambda_2 - \lambda_3) = R_{13}(\lambda_1 - \lambda_3) R_{23}(\lambda_2 - \lambda_3) \check{R}_{12}(\lambda_1 - \lambda_2). \quad (2.10)$$

The role of the  $\check{R}$  is in this sense purely technical: they dress the solutions (2.8) so that the traces (2.7) have non-trivial (non-decoupled) structure, but due to (2.10) they do not modify the exchange relations, which guarantee commutation of (2.7). We emphasize that any object obeying the commutation relation (2.10) is a good dressing operator;  $\check{R}$  is simply an easily constructed example of it.

2.2. *Commuting traces related to reflection algebras*

We now come to our main concern, which is the explicit construction of the commuting traces related to reflection algebras. For this purpose we shall introduce the notions of generalized reflection equations and duals thereof.

We define, in analogy to the case related to Yang–Baxter algebras, the set of generalized reflection equations associated with the fused  $R$ -matrices (2.1) as

$$\begin{aligned} T_{\mathcal{N}q}(\lambda_{\mathcal{N}})R_{\mathcal{M}'\mathcal{N}}(\lambda_{\mathcal{N}} + \lambda_{\mathcal{M}'})T_{\mathcal{M}'q}(\lambda_{\mathcal{M}'})R_{\bar{\mathcal{N}}\mathcal{M}'}(-\lambda_{\bar{\mathcal{N}}} + \lambda_{\mathcal{M}'}) \\ = R_{\mathcal{M}'\bar{\mathcal{N}}}(-\lambda_{\bar{\mathcal{N}}} + \lambda_{\mathcal{M}'})T_{\mathcal{M}'q}(\lambda_{\mathcal{M}'})R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} + \lambda_{\mathcal{M}'})T_{\mathcal{N}q}(\lambda_{\mathcal{N}}) \end{aligned} \tag{2.11}$$

where the objects  $T_{\mathcal{N}q}$  are matrices acting on tensor products of auxiliary spaces indexed by ordered sets  $\mathcal{N}$  as  $V_1 \otimes V_2 \dots \otimes V_n$  with operator entries acting on the ‘bulk’ quantum space labelled by index  $q$ . In general they may also act on some extra boundary quantum space (see e.g. [9, 10]), which is denoted by the index  $q_b$ .

We also introduce the set of generalized ‘dual’ reflection equations, which has the following structure:

$$\begin{aligned} \mathcal{K}_{\mathcal{N}q'}(\lambda_{\mathcal{N}})M_{\mathcal{M}'}R_{\bar{\mathcal{M}}'\bar{\mathcal{N}}}(-\lambda_{\bar{\mathcal{N}}} - \lambda_{\bar{\mathcal{M}'}} - 2\rho)M_{\bar{\mathcal{M}'}}^{-1}\mathcal{K}_{\mathcal{M}'q'}(\lambda_{\mathcal{M}'})R_{\mathcal{N}\bar{\mathcal{M}'}}(\lambda_{\mathcal{N}} - \lambda_{\bar{\mathcal{M}'}}) \\ = R_{\bar{\mathcal{M}}'\mathcal{N}}(\lambda_{\mathcal{N}} - \lambda_{\bar{\mathcal{M}'}})\mathcal{K}_{\mathcal{M}'q'}(\lambda_{\mathcal{M}'})M_{\bar{\mathcal{M}'}}^{-1}R_{\bar{\mathcal{N}}\bar{\mathcal{M}'}}(-\lambda_{\bar{\mathcal{N}}} - \lambda_{\bar{\mathcal{M}'}} - 2\rho)M_{\mathcal{M}'}\mathcal{K}_{\mathcal{N}q'}(\lambda_{\mathcal{N}}). \end{aligned} \tag{2.12}$$

$\mathcal{K}_{\mathcal{N}q'}$  are similarly matrices acting on tensor products of auxiliary spaces indexed by ordered sets  $\mathcal{N}$  as  $V_1 \otimes V_2 \dots \otimes V_n$  with operator entries acting on a *a priori* different ‘bulk’ quantum space labelled by index  $q'$ . In general they may act on some extra boundary quantum space as well (see e.g. [9, 10]), which is denoted by the index  $q'_b$ . At this stage we would like to point out the structural similarity between the generalized reflection equation (2.11), and the corresponding reflection equation for fused  $K$  matrices introduced in [12]. In addition we assume the existence of a transposition antimorphism  $t_{q'q'_b}$  acting on the operator entries of  $\mathcal{K}_{\mathcal{N}q'_b}$ .

We now establish two basic existence lemmas for (2.11) and (2.12).

**Lemma 1** (fusion of generalized reflection matrices). *For simplicity we omit the indices  $q$  and  $q_b$ . If  $T$  is a solution to the reflection equation (1.5) then the following objects:*

$$\begin{aligned} T_{\mathcal{N}}^0 = T_1 R_{21}(\lambda_1 + \lambda_2) R_{31}(\lambda_1 + \lambda_3) \dots R_{n1}(\lambda_1 + \lambda_n) T_2 R_{32}(\lambda_2 + \lambda_3) \dots R_{n2}(\lambda_2 + \lambda_n) \\ \times T_3 \dots T_k R_{k+1k}(\lambda_k + \lambda_{k+1}) \dots R_{nk}(\lambda_k + \lambda_n) T_{k+1} \dots T_{n-1} R_{nn-1}(\lambda_n + \lambda_{n-1}) T_n \end{aligned} \tag{2.13}$$

are solutions to the set of generalized reflection equations (2.11).

**Lemma 2** (fusion of dual generalized reflection matrices). *Again for simplicity we omit the indices  $q'$  and  $q'_b$ . If  $K$  is a solution to the reflection equation (1.9) then the following objects:*

$$\begin{aligned} \mathcal{K}_{\mathcal{N}}^0 = K_n M_{n-1} R_{n-1n}(-\lambda_n - \lambda_{n-1} - 2\rho) M_{n-1}^{-1} K_{n-1} \dots \\ \times K_{k+1} M_k R_{kn}(-\lambda_k - \lambda_n - 2\rho) \dots R_{kk+1}(-\lambda_k - \lambda_{k+1} - 2\rho) M_k^{-1} K_k \dots \\ \times K_2 M_1 R_{1n}(-\lambda_1 - \lambda_n - 2\rho) \dots R_{12}(-\lambda_1 - \lambda_2 - 2\rho) M_1^{-1} K_1 \end{aligned} \tag{2.14}$$

satisfy the dual generalized reflection equation.

Note that the fusion procedure operates exclusively on auxiliary spaces; the quantum spaces  $q, q', q_b, q'_b$  are untouched.

The proofs are established by a recursion procedure on the total number of fused auxiliary spaces in the considered equations:  $n_0 \equiv \text{card}(\mathcal{N} + \mathcal{M}')$ .

**Proof of lemma 1.**

- The lemma is established by hypothesis for  $n_0 = 2$  where necessarily  $\text{card}(\mathcal{N}) = \text{card}(\mathcal{M}') = 1$  and (2.11) reduces to (1.5).
- Assuming now that lemma 1 is proved up to some value  $n_0 \geq 2$  one considers the equations from the set (2.11) at  $n_0 + 1$ . It is always possible to assume that  $\text{card}(\mathcal{N}) \geq 2$ , indeed if not then necessarily  $\text{card}(\mathcal{M}') \geq 2$  and by multiplying the equation from (2.11) on the lhs by  $R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{N}} - \lambda_{\mathcal{M}'})$  and on the rhs by  $R_{\mathcal{M}'\mathcal{N}}(\lambda_{\mathcal{N}} - \lambda_{\mathcal{M}'})$  one gets back, after exchanging the notation  $\mathcal{N}$  and  $\mathcal{M}'$ , a new form of the same equation, this time with  $\text{card}(\mathcal{N}) \geq 2$ .

One then particularizes the first index of the ordered set  $\mathcal{N}$  as 1 and rewrites the equation as (denoting the set  $\mathcal{N} - \{1\}$  as  $\mathcal{N}^-$ ):

$$\begin{aligned} & \mathcal{T}_1(\lambda_1)R_{\mathcal{N}^-1}(\lambda_1 + \lambda_{\mathcal{N}^-})\mathcal{T}_{\mathcal{N}^-}(\lambda_{\mathcal{N}^-})R_{\mathcal{M}'1}(\lambda_1 + \lambda_{\mathcal{M}'})R_{\mathcal{M}'\mathcal{N}^-}(\lambda_{\mathcal{N}'} + \lambda_{\mathcal{M}'}) \\ & \quad \times \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'})R_{\mathcal{N}^-\mathcal{M}'}(-\lambda_{\mathcal{N}^-} + \lambda_{\mathcal{M}'})R_{1\mathcal{M}'}(-\lambda_1 + \lambda_{\mathcal{M}'}) \\ & = R_{\mathcal{M}'\mathcal{N}^-}(-\lambda_{\mathcal{N}^-} + \lambda_{\mathcal{M}'})R_{\mathcal{M}'1}(-\lambda_1 + \lambda_{\mathcal{M}'})\mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'})R_{1\mathcal{M}'}(\lambda_1 + \lambda_{\mathcal{M}'}) \\ & \quad \times R_{\mathcal{N}^-\mathcal{M}'}(\lambda_{\mathcal{N}^-} + \lambda_{\mathcal{M}'})\mathcal{T}_1(\lambda_1)R_{\mathcal{N}^-1}(\lambda_1 + \lambda_{\mathcal{N}^-})\mathcal{T}_{\mathcal{N}^-}(\lambda_{\mathcal{N}^-}). \end{aligned} \tag{2.15}$$

One now establishes the validity of this equality by successive operations on the lhs of (2.15):

1. Using the (already proved by recursion hypothesis) exchange relation for index sets  $\mathcal{N}^-$  and  $\mathcal{M}'$ .
2. Using a fused Yang–Baxter equation:

$$R_{\mathcal{N}^-1}^+ R_{\mathcal{M}'1}^+ R_{\mathcal{M}'\mathcal{N}^-}^- = R_{\mathcal{M}'\mathcal{N}^-}^- R_{\mathcal{M}'1}^+ R_{\mathcal{N}^-1}^+ \tag{2.16}$$

where the compact notations  $R^\pm$  are self-explanatory.

3. Using a second fused Yang–Baxter equation:

$$R_{\mathcal{N}^-1}^+ R_{\mathcal{N}^-\mathcal{M}'}^+ R_{1\mathcal{M}'}^- = R_{1\mathcal{M}'}^- R_{\mathcal{N}^-\mathcal{M}'}^+ R_{\mathcal{N}^-1}^+. \tag{2.17}$$

4. Using the (already proved by recursion hypothesis) exchange relations for sets  $\{1\}$  and  $\mathcal{M}'$ .

Note that the fused Yang–Baxter equations (2.16) and (2.17) are immediate consequences of the coproduct structure of the ordinary Yang–Baxter equation.

This establishes lemma 1. □

**Proof of lemma 2.** The proof is similar, successively applying to the rhs of (2.12) decoupled as in (2.15)

1. The reflection equation for sets  $\mathcal{N}^-$  and  $\mathcal{M}'$ ,
2. the dual fused Yang–Baxter equation

$$R_{\mathcal{N}^-\mathcal{M}'}^+ R_{1\mathcal{M}'}^- R_{1\mathcal{N}^-}^- = R_{1\mathcal{N}^-}^- R_{1\mathcal{M}'}^- R_{\mathcal{N}^-\mathcal{M}'}^+ \tag{2.18}$$

where the notation  $R^{--}$  is also self-explanatory from (2.12)

3. the dual fused Yang–Baxter equation

$$R_{\mathcal{M}'1}^- R_{\mathcal{M}'\mathcal{N}^-}^- R_{1\mathcal{N}^-}^- = R_{1\mathcal{N}^-}^- R_{\mathcal{M}'\mathcal{N}^-}^- R_{\mathcal{M}'1}^- \tag{2.19}$$

4. the reflection equation for  $\{1\}$  and  $\mathcal{M}'$ , and
5. at several places the commutation relation (2.5).

Both lemmas are thus established, hence the set of solutions of (2.11), (2.12) is not empty.

Note that the fused solution  $\mathcal{K}_N^0$  has a structure similar to  $\mathcal{T}_N^0$ , but with a reversed order of the auxiliary spaces; in particular one has an identification of formal solutions as:  $\mathcal{K}_N^0(\lambda_N) \equiv \mathcal{T}_N^0(-\lambda_N - \rho)^{t_N}$ . This is expected because of the form of equations (2.11) and (2.12): equation (2.11) is formally the ‘transpose’ of equation (2.12). The treatment of the general case will follow from the fact that fused YB equations (2.16)–(2.19) take identical forms in the general case once one replaces

$$R_{12}(\lambda_1 - \lambda_2) \rightarrow A \quad R_{21}(\lambda_1 - \lambda_2) \rightarrow D \quad R^+ \rightarrow C = B^\pi. \quad (2.20)$$

This replacement will provide us with a ‘dictionary’ between the reflection algebra structures and the general quadratic structures.

We now establish a very important ‘dressing’ or invariance property of general solutions to the fused equations. It is described by the following proposition.

**Proposition 1** (dressing of solutions).

- (a) Given a set of solutions to the dual generalized reflection equations (2.12) one obtains a new set of solutions by multiplying  $\mathcal{K}_{N,q}$  on the left with operators of the form  $\mathcal{Q}_N$  acting on the sole auxiliary spaces and such that for all sets  $\mathcal{M}'$  disjoint from  $\mathcal{N}$  one has:

$$[\mathcal{Q}_N, R_{\mathcal{M}'\mathcal{N}}(\lambda_N \pm \lambda_{\mathcal{M}'})] = 0 \quad [\mathcal{Q}_N, R_{\bar{\mathcal{N}}\mathcal{M}'}(-\lambda_N \pm \lambda_{\mathcal{M}'})] = 0. \quad (2.21)$$

A similar property will hold for (2.11) under right multiplication by  $\mathcal{Q}_N$ .

- (b) Given a set of solutions to the dual generalized reflection equations (2.12) one obtains a new set of solutions by multiplying  $\mathcal{K}_{N,q}$  on the right with operators of the form  $\mathcal{S}_N$  acting on the sole auxiliary spaces, and such that for all sets  $\mathcal{M}'$  disjoint from  $\mathcal{N}$  one has

$$[\mathcal{S}_N, R_{\mathcal{N}\mathcal{M}'}(\lambda_N \pm \lambda_{\mathcal{M}'})] = 0 \quad [\mathcal{S}_N, R_{\mathcal{M}'\bar{\mathcal{N}}}(-\lambda_N \pm \lambda_{\mathcal{M}'})] = 0. \quad (2.22)$$

A similar property will hold for (2.11) under left multiplication by  $\mathcal{S}_N$ .

The two conditions in (2.21) and (2.22) are equivalent by unitarity of fused  $R$ -matrices.

The proof is immediate from the form of (2.11), (2.12) and the use of (2.5).

We shall discuss for simplicity now only the case of dressing operators of type  $\mathcal{Q}_N$ . Note that in particular the following product of  $\check{R}$  defined by (2.10):

$$\mathcal{Q}_N = \check{R}_{12}(\lambda_1 - \lambda_2) \dots \check{R}_{n-1n}(\lambda_{n-1} - \lambda_n) \quad (2.23)$$

realizes such a dressing. As indicated in section 2.1. the construction of  $\check{R}$  as  $P_{12}R_{12}$  is problematic when considering spectral-parameter dependent  $R$  matrices where the permutation operator must act on a loop space  $(V_1 \otimes C(\lambda_1)) \otimes (V_2 \otimes C(\lambda_2))$ . One may, however, establish at the level of formal series a better defined solution.

**Proposition 2.**

$$\check{R} = \delta(\lambda_1, \lambda_2) \Pi_{12} R_{12} \quad (2.24)$$

realizes (2.10) as formal series, where  $\Pi_{12}$  is the operator exchanging vector spaces  $V_1, V_2$ .

Here the  $\delta$  distribution is of course defined as a formal series

$$\delta(\lambda_1, \lambda_2) = \sum_{n \in \mathbb{Z}} (\lambda_1/\lambda_2)^n. \quad (2.25)$$

The proof follows from the formal series identity:

$$\delta(\lambda_1, \lambda_2) f(\lambda_1) g(\lambda_2) = f(\lambda_2) g(\lambda_1) \delta(\lambda_1, \lambda_2) \quad (2.26)$$



for any functions  $f, g$  with an assumed Laurent series expansion defined as

$$f(\lambda) = \sum_{-\infty < m_0 < m} f_m \lambda^m \quad g(\lambda) = \sum_{-\infty < p_0 < p} g_p \lambda^p. \tag{2.27}$$

The explicit proof follows from comparing both sides of (2.26) as formal series: the coefficients of  $\lambda_1^a \lambda_2^b$  on both sides are

$$\sum_{m, p | m+p=a+b} (f_m g_p) \tag{2.28}$$

which is a finite sum by the Laurent series hypothesis on  $f, g$  ( $-\infty < m_0, -\infty < p_0$ ). Provided that one deals with  $R$  matrices expandable as Laurent series, one then immediately deduces that (2.24) satisfies (2.10), and hence (2.23) satisfies (2.21).

As in the Yang–Baxter-type algebras, having determined the proper generalized exchange relations, we are now in a position to build commuting traces. We establish the fundamental theorem.

**Theorem.** *Let  $\mathcal{K}_{\mathcal{N}}$  be a set of solutions to the dual generalized reflection equations (2.12), acting on the auxiliary spaces labelled by  $\mathcal{N}$ , the quantum space labelled by  $q'$ , and a possible boundary space labelled by  $q'_b$ ;*

*Let  $\mathcal{T}_{\mathcal{N}}$  be a set of solutions of the generalized reflection equations (2.11) acting on the tensor product of the auxiliary spaces labelled by  $\mathcal{N}$ , the quantum space labelled by  $q$ , and the boundary space labelled by  $q_b$ .*

*The following trace operators acting on the quantum space  $q \otimes q_b \otimes q' \otimes q'_b$ :*

$$H_{\mathcal{N}} = \text{Tr}_{\mathcal{N}}(\mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}})) \tag{2.29}$$

where  $\mathcal{K}_{\mathcal{N}}^* = \mathcal{K}_{\mathcal{N}}^{t'q'_b}$ , build a family of mutually commuting operators:

$$[H_{\mathcal{N}}, H_{\mathcal{M}'}] = 0. \tag{2.30}$$

**Proof.** The proof is identical to the proof of (1.8) (see [6]) thanks to the fusion relations of lemma 1 and 2; it is however worth being given in detail. Quantum indices will again be dropped for simplicity.

One starts from the product  $H_{\mathcal{M}'} H_{\mathcal{N}}$ ; the idea is to pass the unprimed indices of the product through the primed indices. For this purpose we first act by the partial transposition  $t_{\mathcal{M}'}$ , yielding

$$\begin{aligned} H_{\mathcal{M}'} H_{\mathcal{N}} &= \text{tr}_{\mathcal{M}'} \mathcal{K}_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'}) \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'}) \text{tr}_{\mathcal{N}} \mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}) \\ &= \text{tr}_{\mathcal{M}'} \mathcal{K}_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \text{tr}_{\mathcal{N}} \mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}) \\ &= \text{tr}_{\mathcal{M}'\mathcal{N}} \mathcal{K}_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}). \end{aligned} \tag{2.31}$$

We now use the crossing-unity of the  $R$  matrix

$$M_{\mathcal{M}'}^{-1} R_{\mathcal{M}'\mathcal{N}}(-\lambda_{\mathcal{M}'} - \lambda_{\mathcal{N}} - 2\rho)^{t_{\mathcal{M}'}} M_{\mathcal{M}'} R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}})^{t_{\mathcal{M}'}} = Z(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \tag{2.32}$$

where  $Z$  is just a function of  $\lambda$ . Then the product  $H_{\mathcal{M}'} H_{\mathcal{N}}$  becomes

$$\begin{aligned} &Z^{-1}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \text{tr}_{\mathcal{M}'\mathcal{N}} \mathcal{K}_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}) M_{\mathcal{M}'}^{-1} R_{\mathcal{N}\mathcal{M}'}(-\lambda_{\mathcal{M}'} - \lambda_{\mathcal{N}} - 2\rho)^{t_{\mathcal{M}'}} M_{\mathcal{M}'} \\ &\quad \times R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}})^{t_{\mathcal{M}'}} \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}) \\ &= Z^{-1}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \text{tr}_{\mathcal{M}'\mathcal{N}} (\mathcal{K}_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} M_{\mathcal{M}'}^{-1} R_{\mathcal{N}\mathcal{M}'}(-\lambda_{\mathcal{M}'} - \lambda_{\mathcal{N}} - 2\rho) \\ &\quad \times M_{\mathcal{M}'} \mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}}))^{t_{\mathcal{M}'}} (\mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'}) R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}))^{t_{\mathcal{M}'}} \\ &= Z^{-1}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \text{tr}_{\mathcal{M}'\mathcal{N}} (K_{\mathcal{M}'}^*(\lambda_{\mathcal{M}'})^{t_{\mathcal{M}'}} M_{\mathcal{M}'}^{-1} R_{\mathcal{N}\mathcal{M}'}(-\lambda_{\mathcal{M}'} - \lambda_{\mathcal{N}} - 2\rho) \\ &\quad \times M_{\mathcal{M}'} K_{\mathcal{N}}^*(\lambda_{\mathcal{N}})^{t_{\mathcal{M}'}})^{t_{\mathcal{M}'\mathcal{N}}} \mathcal{T}_{\mathcal{M}'}(\lambda_{\mathcal{M}'}) R_{\mathcal{N}\mathcal{M}'}(\lambda_{\mathcal{M}'} + \lambda_{\mathcal{N}}) \mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}). \end{aligned} \tag{2.33}$$

Using the unitarity of the  $R$  matrix

$$R_{\bar{N},\bar{M}'}(-\lambda_{\bar{M}'} + \lambda_{\bar{N}})R_{M',N}(\lambda_{M'} - \lambda_N) = Z(\lambda_{M'} - \lambda_N) \tag{2.34}$$

we obtain the following expression for the product:

$$\begin{aligned} & Z^{-1}(\lambda_{M'} - \lambda_N)Z^{-1}(\lambda_{M'} + \lambda_N) \operatorname{tr}_{M',N}(\mathcal{K}_{M'}^*(\lambda_{M'})^{t_{M'}} M_{M'}^{-1} \\ & \quad \times R_{N,M'}(-\lambda_{M'} - \lambda_N - 2\rho)M_{M'}\mathcal{K}_{N'}^*(\lambda_{N'})^{t_{N'}}^{t_{M',N}} \\ & \quad \times R_{N,\bar{M}'}(-\lambda_{\bar{M}'} + \lambda_N)R_{M',\bar{N}}(\lambda_{M'} - \lambda_{\bar{N}})\mathcal{T}_{M'}(\lambda_{M'})R_{N,M'}(\lambda_{M'} + \lambda_N)\mathcal{T}_{N'}(\lambda_N) \\ = & Z^{-1}(\lambda_{M'} - \lambda_N)Z^{-1}(\lambda_{M'} + \lambda_N) \operatorname{tr}_{M',N}(R_{M',\bar{N}}(-\lambda_{M'} + \lambda_{\bar{N}})\mathcal{K}_{M'}^*(\lambda_{M'})^{t_{M'}} \\ & \quad \times M_{M'}^{-1}R_{N,M'}(-\lambda_{M'} - \lambda_N - 2\rho)M_{M'}\mathcal{K}_{N'}^*(\lambda_{N'})^{t_{N'}}^{t_{M',N}} \\ & \quad \times R_{M',\bar{N}}(\lambda_{M'} - \lambda_{\bar{N}})\mathcal{T}_{M'}(\lambda_{M'})R_{N,M'}(\lambda_{M'} + \lambda_N)\mathcal{T}_{N'}(\lambda_N). \end{aligned} \tag{2.35}$$

One now recognizes on the rhs: (1) the rhs of the exchange equation (2.11); (2) the full transposition under  $t_{N,M'}t_{q'q'_b}$  of the lhs of the exchange relation (2.12) (recall (2.2)). The specific forms of unitarity (2.34) and crossing-unitarity (2.32) are crucial in yielding this form. This hints at a close connection between the crossing-unitarity properties and the duality ‘transformation’ between (2.11) and (2.12).

Now, with the help of equations (2.11), (2.12), (2.32) and (2.34) and by repeating the previous steps in reverse order we establish that the last expression is indeed  $H_N H_{M'}$ . This concludes the proof of the commutativity relation (2.30).  $\square$

Note that it is needed at the very beginning of this proof to use mutual commutation of the matrix elements of  $\mathcal{K}$  and  $\mathcal{T}$ . We then emphasize that the above proof is valid as long as neither the left and right boundary spaces  $q_b$  and  $q'_b$ , nor the quantum spaces  $q$  and  $q'$  ‘talk’ to each other.

### 2.3. Dressed quantum traces

The dressing procedure of fused matrices was defined by proposition 1. It generically leads to traces of the form  $\operatorname{Tr}(\mathcal{Q}\mathcal{K}^0\mathcal{S}\mathcal{T}^0)$ . We shall again consider only at this stage dressing by objects of type  $\mathcal{Q}$  leading to traces of the form  $\operatorname{Tr}(\mathcal{Q}\mathcal{K}^0\mathcal{T})$  so as to avoid complicating the discussion.

This construction is needed to get non-trivial commuting traces. Indeed, it is easy to prove:

**Proposition 3** (factorized quantum traces). *Operators built from the basic fused solutions (2.13), (2.14) decouple as  $H_N = \operatorname{Tr}_N(\mathcal{K}_N^{0*}\mathcal{T}_N^0) = (\operatorname{Tr}_1 K_1^* T_1)^{\operatorname{card} N}$ .*

The proof is achieved by successive use of partial transpositions with respect to successive indices of  $\mathcal{N}$  from 1 to  $n$  as defined in (2.13) and (2.14). These partial transpositions systematically bring together fused transposed  $R$  matrices which cancel each other by fused crossing relations (2.4). Hence the complete elimination of the  $R$  matrices, yielding  $\operatorname{Tr}_N(\mathcal{K}_N^{0*}\mathcal{T}_N^0) = \operatorname{Tr}((K_n \dots K_1)^* T_1 \dots T_n)$ , then by using the antimorphism property of the  $*$  operation one obtains  $\operatorname{Tr}(K_1^* \dots K_n^* T_1 \dots T_n) = (\operatorname{Tr} K_1^* T_1)^n$ . Such was also the more obvious case for the usual Yang–Baxter algebras (2.7), (2.8).

Therefore additional non-trivial dressing operators are required to get non-trivial traces. As indicated by proposition 2, examples of such objects are already formally available as products of neighbouring-indexed  $\delta(\lambda_a, \lambda_{a+1})\Pi_{a,a+1}R_{a,a+1}$ . This particular choice of dressing deserves a more detailed discussion. It actually provides an interesting classical limit. We

here assume that the classical limits are defined in the usual way, i.e.

$$R_{12}(\lambda_1 - \lambda_2) = \mathbf{1} \otimes \mathbf{1} + \hbar r_{12}(\lambda_1 - \lambda_2) \quad T(\lambda) = t(\lambda) + o(\hbar) \quad K(\lambda) = 1 + o(\hbar). \quad (2.36)$$

One then easily shows that

**Proposition 4.** *The coefficients of the operators  $H_{\mathcal{N}} = \text{Tr}_{\mathcal{N}}(\mathcal{K}_{\mathcal{N}}^*(\lambda_{\mathcal{N}})\mathcal{T}_{\mathcal{N}}(\lambda_{\mathcal{N}}))$  expanded as a multiple formal series with general term  $\lambda_1^{a_1}, \dots, \lambda_n^{a_n}$ ,  $n = \text{card}(\mathcal{N})$  are identical, when  $\hbar$  goes to zero, to the coefficients of the classical trace  $h_n = \text{Tr}(t^n(\lambda))$  expanded as a formal series with general term  $\lambda^m$  with the identification  $m = \sum_{i=1}^n a_i$ .*

**Proof.** The properties of the permutation operators  $\Pi_{a,a+1}$  inside the multiple trace reduce it to a single trace of direct products, once both  $R$  and  $K$  factors are taken to identity. The formal series expansion of the product of  $\delta$  distributions then yields the suggested identification at  $m = \sum_{i=1}^n a_i$  of the coefficients of the two formal series. Again we assume that  $t(\lambda)$  can be formally expanded as a Laurent series.

Since the classical limit of these particular dressed traces yields the classical trace of a power of the classical Lax matrix we are justified, as was the case in [1], in denoting these operators as ‘quantum traces’, *a priori algebraically distinct* from operators obtained from powers of the commuting traces of the quantum Lax matrix.

Let us conclude this brief discussion by a general remark. The problem of constructing generators such that  $[\mathcal{Q}_{\mathcal{N}}, R_{\mathcal{N},\mathcal{M}}] = 0$  for general fused  $R$  matrices is in any case related to the understanding of the underlying coproduct structure and universal algebra. In this context  $\mathcal{Q}$  objects may be constructed generically as coproducts of central elements<sup>4</sup>.  $\square$

### 3. Conclusion and prospective

In order to define lines of future investigation it is important to characterize the basic steps of the procedure described in the previous section. It crucially depends on four fundamental features:

*Step 1.* Existence of an algebraic reflection-like structure (1.5) with the notion of a dual structure (1.9); associated unitarity and crossing relations, and dual trace formula generating commuting objects as in (1.7).

*Step 2.* Existence of mutually consistent fusion procedures for both algebraic structures as described in lemma 1 and 2.

*Step 3.* Dressing of fused solutions by commuting (fused) operators  $\mathcal{Q}$  on the auxiliary spaces, such as characterized by proposition 1.

*Step 4.* One then combines 1, 2 and 3 to get commuting traces by products of fused solutions  $\mathcal{T}$  and  $\mathcal{K}$ .

This now indicates several directions of investigation:

#### 3.1. Extension of the procedure to general reflection algebras of (1.4) type

Interest in this generalization stems in particular from the occurrence in physical systems of relevant examples of braided YB algebras: the structure matrices  $A, B, C, D$  are in general not independent or free. For instance, the reflection algebra itself is characterized by  $A_{12} = D_{21} = R_{12}(\lambda_1 - \lambda_2)$  and  $B_{12} = C_{21} = R_{12}(\lambda_1 + \lambda_2)$ . Different choices of constraints

<sup>4</sup> We are indebted to Daniel Arnaudon for this suggestion.

realize different algebraic objects. Such are for instance the reflection–transmission algebras [15] where  $A, B, C, D$  are given by one single quantum  $R$  matrix depending however on two independent spectral parameters instead of one single combination  $\lambda_1 \pm \lambda_2$ ; the non-ultralocal monodromy matrix algebra for mKdV equations [16, 17] where the  $B$  matrices are diagonal  $c$ -number matrices; and cases where the  $R$  matrix does not necessarily satisfy the crossing symmetry (1.11), but a more general form of crossing (see e.g. [18]).

The extension will be undertaken [7] by following steps 1–4 as previously defined.

Step 1 was actually realized in [4] where a dual reflection algebra was defined. It is indeed easy to check that it gives back (1.9) in our case once one uses unitarity, crossing and transposition relations (1.11) and (1.10). The commuting traces are again of the form (1.7). However in [4] the matrices  $\mathcal{K}$  are taken to be pure  $c$ -number objects, not quantum operators; hence it remains to prove consistency of the definition for quantum  $\mathcal{K}$  matrices.

The next problem is to realize step 2, defining fusion procedures for quadratic algebras generalizing lemmas 1 and 2. Fusion procedures are obtained easily once a coproduct structure has been formulated; however consistent fusion procedures may exist at a represented level without being direct consequences of underlying universal coproduct structures: for instance it seems to be the case for the formulation given here<sup>5</sup>.

By contrast another fusion procedure exists, compatible with (1.5), defined in [20] as following from a universal coproduct structure. Connections between these two fusion processes will be discussed in detail in the general context. Briefly, the basic ingredient in this connection is a ‘braiding’ matrix  $\mathcal{F}_{\mathcal{N}}$ , i.e. acting on fused  $R$ -matrices as:

$$\mathcal{F}_{\mathcal{N}} R_{\mathcal{N}\mathcal{M}'} = R_{\tilde{\mathcal{N}}\mathcal{M}'} \mathcal{F}_{\mathcal{N}}. \quad (3.1)$$

This procedure can naturally also be used to define quantum traces from (2.29). The comparison between the two constructions will be left for general discussion.

It is thus possible to define *several* fusion procedures and trace formulae, depending on the considered subclasses of quadratic algebras, characterized in particular by different restrictions on the structure matrices  $A, B, C, D$ .

In view of physical applications such as already mentioned, it is in fact very interesting to examine which restrictions on  $A, B, C, D$  are compatible with the duality construction and how they relate to particular crossing and unitarity conditions. This in turn will need a careful analysis of the discrete (possibly infinite) symmetry groups of the braided YB equations analogous to the studies conducted e.g. in [19].

Finally we have already discussed the extensions of step 3.

### 3.2. Extensions to quadratic algebras of dynamical type

Here the matrices  $R$  and  $K$  should depend on some extra ‘dynamical’ parameters identified as coordinates on the dual of some (possibly non-Abelian, see [21]) Lie algebra, and the exchange relations take a form along the lines of the dynamical Yang–Baxter equation described in [21, 22]. The first problem at this time is to actually construct such extensions. Some examples are currently being considered [23, 24]. Quantum trace formulae may then be obtained by suitably manipulating our construction along the lines in [25] for the dynamical Yang–Baxter equation, so as to incorporate the shifted dynamical parameter. Again one essentially needs to define consistent fusion procedures for such algebras and consistent unitarity and crossing properties.

In particular, if one obtains solutions of the example of dynamical quantum reflection equation defined in [23], which incorporates as a structure matrix Felder’s dynamical

<sup>5</sup> As follows from our enlightening discussions with P P Kulish.

$R$ -matrix [22], one expects to be able to define generalizations of the Ruijsenaars–Schneider Hamiltonians [26] obtained in [25]. Such generalizations, incorporating non-trivial reflection algebras, may be connected with the quantum Hamiltonians presented e.g. in [27–30] as the construction [29] suggests, however, more general integrable Hamiltonians may also arise.

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### References

- [1] Maillet J M 1990 Lax operators and quantum groups *Phys. Lett. B* **245** 481
- [2] Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys.* **70** 193  
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [3] Korepin V E 1980 New effects in the massive Thirring model, repulsive regime *Commun. Math. Phys.* **76** 165  
Korepin V E, Izergin G and Bogoliubov N M 1993 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)
- [4] Freidel L and Maillet J M 1991 Quadratic algebras and integrable systems *Phys. Lett. B* **262** 278
- [5] Cherednik I 1984 Factorizing particles on half-line and root systems *Theor. Math. Phys.* **61** 977
- [6] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375
- [7] Avan J, Doikou A and Nagy Z in preparation
- [8] Arnaudon D, Avan J, Crampé N, Frappat L and Ragoucy E 2003  $R$  matrix representation for super Yangians  $\mathcal{Y}(g)$  *J. Math. Phys.* **44** 302
- [9] Baseilhac P and Koizumi K 2003 Sine-Gordon quantum field theory on the half-line with quantum boundary degrees of freedom *Nucl. Phys. B* **49** 491
- [10] Doikou A and Martin P P 2003 Hecke algebraic approach to the reflection equation for spin chains *J. Phys. A: Math. Gen.* **36** 2203
- [11] Arnaudon D, Avan J, Crampé N, Doikou A, Frappat L and Ragoucy E 2003 Classification of reflection matrices related to (super) Yangians and applications to open spin chains *Nucl. Phys. B* **668** 469
- [12] Mezincescu L and Nepomechie R I 1992 Fusion procedure for open chains *J. Phys. A: Math. Gen.* **25** 2533  
Mezincescu L, Nepomechie R I and Rittenberg V 1990 Bethe ansatz solution of the Fateev–Zamolodchikov quantum spin chain with boundary terms *Phys. Lett. A* **147** 70
- [13] Drinfeld V G 1986 Quantum groups *Proc. Int. Congress of Mathematicians (Berkeley 1987)* ed A V Gleason (Providence, RI: American Mathematical Society) p 78
- [14] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 Yang–Baxter equation and representation theory *Lett. Math. Phys.* **5** 393
- [15] Mintchev M, Ragoucy E and Sorba P 2003 Reflection–transmission algebras *J. Phys. A: Math. Gen.* **36** 10407
- [16] Kundu A 1995 Exact Bethe ansatz solution of non ultralocal quantum mKdV model *Mod. Phys. Lett. A* **10** 2955  
Hlavaty L 1994 Quantum braided groups *J. Math. Phys.* **35** 2560
- [17] Fioravanti D and Rossi M 2002 A braided Yang–Baxter algebra in a theory of two coupled lattice quantum KdV *J. Phys. A: Math. Gen.* **35** 3647–82
- [18] Doikou A 2000 Quantum spin chain with ‘soliton non-preserving’ boundary conditions *J. Phys. A: Math. Gen.* **33** 8797
- [19] Bellon M, Maillard J M and Viallet C M 1991 On the symmetries of integrability *Phys. Rev. Lett.* **67** 1373
- [20] Donin J and Mudrov A I 2002 Reflection equation, twist and equivalent quantization *Isr. J. Math.* **136** 11  
Donin J, Kulish P P and Mudrov A I 2002 On universal solutions to reflection equation *Lett. Math. Phys.* **63** 179
- [21] Xu P 2002 Quantum dynamical Yang–Baxter equation over a non-abelian base *Commun. Math. Phys.* **226** 475
- [22] Gervais J L and Neveu A 1984 Novel triangle relations and absence of tachyons in Liouville string field theory *Nucl. Phys. B* **238** 125  
Felder G 1984 Conformal field theory and integrable systems associated to elliptic curves *Proc. ICMP (Paris)* vol 211 (Preprint hep-th/9407154)

- 
- [23] Nagy Z, Avan J and Rollet G 2004 *Preprint* math-QA 0307026 *Lett. Math. Phys.* to appear
- [24] Ragoucy E private communication
- [25] Avan J, Babelon O and Billey E 1996 The Gervais–Neveu–Felder equation and quantum Calogero–Moser systems *Commun. Math. Phys.* **178** 281
- [26] Ruijsenaars S N M and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370
- Ruijsenaars S N M 1987 Complete integrability of relativistic Calogero–Moser systems *Commun. Math. Phys.* **110** 191
- [27] Koornwinder T H R 1992 Askey–Wilson polynomials for root systems of type BC *Contemp. Math.* **138** 189
- [28] van Diejen J F 1995 Commuting difference operators with polynomial eigenfunctions *Comput. Math.* **95** 183
- [29] Komori Y 1997 Quantum integrability of the generalized elliptic Ruijsenaars model *J. Phys. A: Math. Gen.* **30** 4341
- Komori Y 1998 Affine  $R$ -matrix and the generalized elliptic Ruijsenaars model *J. Math. Phys.* **35** 6175
- Hikami K and Komori Y 1998 Diagonalization of elliptic Ruijsenaars model *Europhys. J. B* **5** 583
- [30] Avan J and Rollet G 2002  $C_N^+ 1^{(2)}$  Ruijsenaars–Schneider models *Lett. Math. Phys.* **60** 177